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Topological Dynamics of Flipping Lorentz Lattice Gas Models

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We study the topological dynamics of the flipping mirror model of Ruijgrok and Cohen with one or an infinite number of particles. In particular we prove the topological transitivity and topological mixing up to a natural first integral for the one-particle model.

KEY WORDS: Topological dynamics; topological transitivity; periodic points; topological mixing; topological entropy; lattice gas; Lorentz gas; wind-tree model.

1. INTRODUCTION

We consider two-dimensional lattice versions of the Lorentz gas[(1)] or the Ehrenfest wind-tree model.[(2)] Several types of Lorentz lattice gas models have been introduced in the physics literature. The models fall into two groups: probabilistic and deterministic. The deterministic models have infinite memory and thus differ from the probabilistic models, which are Markovian in nature. The models have been intensively studied from a numerical point of view and exhibit rich and varied behavior.[(3–9)] In particular, the physics literature is interested in the asymptotic distribution of the position and velocity of the particle. The deterministic model which is the best candidate for having asymptotically Gaussian distribution is the flipping mirror model of Ruijgrok and Cohen.[(3)] In this paper we study the topological dynamics of this model. Our main result is that this model is topologically mixing (up to a first integral in certain cases). We show that the topological entropy is infinite and that periodic points are dense. We

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also study the flipping mirror model with an infinite number of particles. Here we show that the topological entropy is infinite and that periodic points are dense. Finally we show that both the one- and infinite-particle models are not expansive.

2. DESCRIPTION OF THE MODELS

In the flipping mirror model (FM) two-sided mirrors, interpreted as scatterers or trees, are placed at the sites of the square lattice $\mathbf{Z}^2$. They can align along either one of the diagonal directions of the lattice, and will be called left or right mirrors, depending on the direction (Fig. 1). The mirrors are placed in all possible configurations Ω_m on $\mathbf{Z}^2$, that is, $\Omega_m = \{L, R\}^{\mathbf{Z}^2}$. We will also consider sometimes the configuration space Ω_0 in which not all lattice sites have mirrors, that is, $\Omega_0 = \{\emptyset, L, R\}^{\mathbf{Z}^2}$. A single particle with unit speed and four possible directions propagates with unit speed along the bonds of the lattice and is reflected by the scatterers. A mirror changes from left to right and vice versa each time it is hit by a particle (Fig. 2).

Let us introduce a topology on our process. We always think of the lattice site where the particle is located as the origin and label only the compass direction in which the particle travels. Thus $X_i := \Omega_i \times \{N, S, E, W\}$ are the phase spaces of the FM model ($i \in \{0, m\}$). Let $\pi_1 : X_i \to \Omega_i$ and $\pi_2 : X_i \to \{N, S, E, W\}$ be the natural projections. For $x, y \in X_0$ (resp. X_m) we define a distance by $d(x, y) = 1$ if $\pi_2(x) \neq \pi_2(y)$ and if $\pi_2(x) = \pi_2(y)$ then by the equation $d(x, y) = 3^{-n^2}$ [resp. $d(x, y) = 2^{-n^2}$] if $\pi_1(x)_{i,j} = \pi_1(y)_{i,j}$ for all i, j satisfying $\max(|i|, |j|) < n$ and $\pi_1(x)_{i,j} \neq \pi_1(y)_{i,j}$ for some i, j with $i = n$ or $j = n$. Let $f: X_i \to X_i$ be the FM transformation. The topology where we additionally remember the location of the particle is not natural from the point of view of topological dynamics since it is not compact and from the point of view of ergodic theory since any nonatomic invariant probability measure must be defined on the topology induced by the metric d. Additionally the dynamics of the FM model is not interesting in this topology since all orbits are unbounded.[10]

Fig. 1. Right mirror and left mirror.

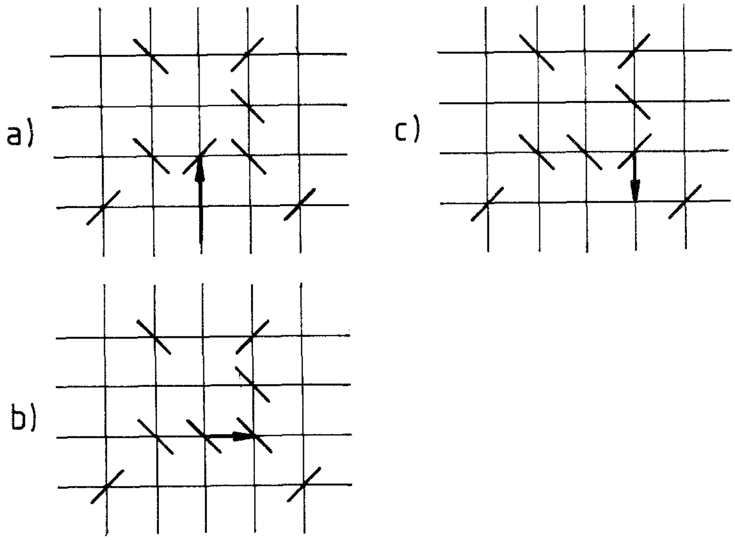

Fig. 2. The particle's motion.

We also consider the flipping mirror model with more than one particle. For a finite number of particles there is no natural compact topology, so we turn to an infinite number of particles (FM_∞). As long as it does not encounter any other particle, the dynamics of each particle is the same as in the FM model. If two particles arrive at the same location at the same time, they simply pass through each other. Particles with the same direction cannot occupy a single lattice site. Mirrors flip with the following rules: if an odd number of particles hit the mirror at the same time, then the mirror flips, and if an ever number of particles hit, then the mirror does not flip.

Let P be the set of all subsets of $\{N, S, E, W\}$. Note that the cardinality of the set P is 16. The set $Z_i := \Omega_i \times P^{\mathbf{Z}^2}$ is the configuration space of the FM_∞ model. Again π_1, π_2 are the natural projections. For $x, y \in Z_0$ (resp. Z_m) we define the distance $d_\infty(x, y)$ to be $(19)^{-n^2}$ [resp. $(18)^{-n^2}$] if $x_{i,j} = y_{i,j}$ for all i, j satisfying $\max(|i|, |j|) < n$ and $x_{i,j} \neq y_{i,j}$ for some i, j with $i = n$ or $j = n$.

3. STATEMENT OF THEOREMS

First we introduce the relevant definitions from topological dynamics. Details can be found in ref. 12 or in any standard text on topological dynamics or ergodic theory.

Definition. A homeomorphism T of a compact metric space (M, d) is called a *cascade.*

Definition. A cascade (M, T) is called *topologically transitive* if there exists $z \in M$ such that $\{T^n z\colon n \in \mathbb{Z}\}$ is dense in M. A topologically transitive cascade is sometimes called topologically ergodic.

Definition. A cascade (M, T) is called *topologically mixing* if, given nonempty open $U, V \subset M$, there is $n_0 \in \mathbb{N}$ such that $T^n U \cap V \neq \varnothing$ whenever $n \geqslant n_0$.

Note that the usual measure-theoretic notions of ergodicity and mixing with respect to any invariant Borel measure imply the respective topological notions.

Definition. A cascade is called *expansive* if there exists a $\delta > 0$ such that if $x \neq y$, then $d(T^n x, T^n y) > \delta$ for some $n \in \mathbb{Z}$.

Two points $x, y \in M$ are said to be *(n, ε)-separated* if $d(T^k x, T^k y) > \varepsilon$ for some $k = 0, 1, \ldots, n-1$. A set $E \subset M$ is called *(n, ε)-separated* if x and y are (n, ε)-separated whenever $x, y \in E$ and $x \neq y$. Then the maximum number of distinguishable orbit n-blocks is $s(n, \varepsilon) := \max\{\text{card } E\colon E \subset M \text{ is } (n, \varepsilon)\text{-separated}\}$. Next we define $h(T, \varepsilon) := \limsup_{n \to \infty} (1/n) \log_2 s(n, \varepsilon)$.

Definition. The *topological entropy* of the cascade (M, T) is defined to be $h_{\text{top}}(T) := \lim_{\varepsilon \to 0^+} h(T, \varepsilon)$.

For the rest of the paper whenever we make a statement about the set X (resp. Z) we mean that the statement is true for both X_0 (resp. Z_0) and X_m (resp Z_m). All measure-theoretic statements will be with respect to any product measure on X or Z which gives all cylinder sets positive mass. We consider first the one-particle model. The following theorem should be contrasted with the fact that all orbits of the FM model are unbounded on $\mathbf{Z}^2$.[10]

Theorem 1. I. The FM model $f\colon (X, d) \to (X, d)$ satisfies the following:

(a) f is topologically transitive

(b) f is not expansive

(c) periodic points are dense, uncountable, and have zero measure

(d) $h_{\text{top}}(f) = \infty$

II. The FM model $f\colon (X_0, d) \to (X_0, d)$ is topologically mixing.

III. For the FM model $f\colon (X_m, d) \to (X_m, d)$ the set X_m can be decomposed into two subsets $A \cup B$ such that $fA = B$ and $fB = A$ and $f^2 | A$ and $f^2 | B$ are topologically mixing.

Next we consider the infinite-particle model. We need one more definition.

Definition. A configuration $x \in Z$ is called *strongly periodic* if x is periodic and the orbit of each individual particle is closed in (Z, d_∞).

Theorem 2. I. The FM_∞ model f: $(Z, d_\infty) \to (Z, d_\infty)$ satisfies the following:

(a) periodic points are dense and uncountable

(b) the set of strongly periodic points is uncountable

(c) f is not expansive

II. The FM_∞ model f: $(Z_0, d_\infty) \to (Z_0, d_\infty)$ has infinite topological entropy.

The topological transitivity and mixing as well as the infinite entropy of the full occupancy case of the FM_∞ model remain open. Both Theorems 1 and 2 also hold for the flipping rotator model of Gunn and Ortuño in the full occupancy case.[9,10] The proofs are completely analogous to those given below.

4. PROOF OF THEOREM 1

First we make some more definitions which will be used in the proofs. Let $\Sigma_M := \{0, 1,..., M-1\}^{\mathbb{Z}}$ be the set of bi-infinite sequences with an M symbol alphabet. Let $\sigma: \Sigma_M \to \Sigma_M$ be the shift mapping defined by $\sigma(\{x_n\}) := \{y_n\}$, where $y_n = x_{n+1}$. The dynamical system (Σ_M, σ) is called the full shift on M symbols. It is well known that the topological entropy of (Σ_M, σ) is $\log M$.[12]

In the topology induced by the metric d the lattice site at which the particle is located is always called the origin; however, for the proof it is often convenient to think of the particle as starting at the origin and then traveling on $\mathbf{Z}^2$. By the trail of x we mean the subset of $\mathbf{Z}^2$ which $f^i x$ hits ($i \in \mathbb{Z}$). The key point to parts (b), (c), and (d) of Theorem 1 is the fact that for periodic points the trail of x is contained in a finite-width strip in $\mathbf{Z}^2$.

Let $G_I := \{(i, j) \in \mathbf{Z}^2\colon \max(|i|, |j|) \leqslant I\}$. The set G_I is called a box. Finally, let $\tilde{f} := f^{-1}$.

I. (a) The topological transitivity follows from topological mixing in II, III.[12]

(b) Consider the configuration x defined as follows: $\pi_1(x)_{i,j} = R$ iff $i = j$ and $i \geqslant 1$ or $i = j - 1$ and $i \geqslant 0$. The other lattice site have left mirrors and $\pi_2(x) = N$ (see Fig. 3). The configuration x is a periodic point of period

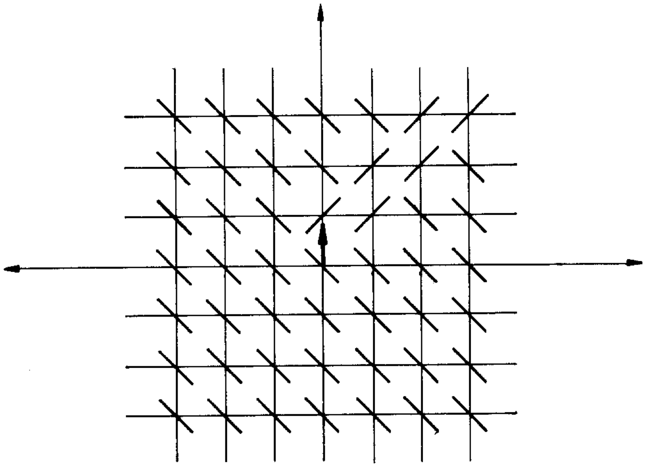

Fig. 3. A periodic point with period 2.

2. Now, given $\delta > 0$, fix I so large that $3^{-I^2} < \delta$ and $I > 2$. For x above $(I, -I) \notin \text{trail}(x)$. Thus we can construct a new configuration $x^* \in X$ which is identical to x except that $\pi_1(x^*)_{I,-I} = R$. The points x and x^* are distinct, yet their orbits are always closer than δ. Note that the point x^* is not a periodic point, but is asymptotically periodic in both forward and backward time.

(c) The simplest periodic point x was given in part (b). All periodic points have a similar feature: the configuration has three parts, a periodic "past" (the backward trail of f), a periodic "future" (the forward trail of f), and the "present." To see that periodic points are dense, fix a configuration x and $\delta > 0$ small. We construct a new configuration y which is periodic and within a distance δ (in the metric d) of the point x. Fix I so large that $3^{-I^2} < \delta$. Let x_I be the restriction of the configuration x to the box G_I. The particle will leave the box under forward iterations of f and also of $\tilde{f}$.[10] Here leaving the box for f means that the particle is in the box but pointing out of the box and for $\tilde{f}$ that the particle is outside of the box (and pointing into the box). We can assume without loss of generality that under f the particle leaves G_I from the top and under $\tilde{f}$ it leaves from the bottom (if it does not, we can increase I slightly and by filling in the extra lattice sites in a correct manner, that is, by two nonintersecting simple paths (i.e., no self-intersection), one for f and one for $\tilde{f}$, we can direct the particle out the top and bottom, respectively). Let $x_I^+ := f^{n_1} x_I$ and $x_I^- := \tilde{f}^{n_2} x_I$, where n_1 and n_2 are the first departure times (as defined above) from the box G_I. Let $\mathbf{a}^+$ and $\mathbf{a}_-$ be the location (lattice site) of the particle in x_I^+ and x_I^-, respectively, and $\mathbf{a} := \mathbf{a}^+ - \mathbf{a}^-$. We now are ready to define the configuration y. The configuration $y \mid G_I$ will be x_I; this is the "present" of the configuration. The "future" will be the periodic repetition of $\pi_1(x_I^-)$ on the boxes $G_I + n \cdot \mathbf{a}$ for all $n > 0$, and the "past" will be the periodic repetition of $\pi_1(x_I^+)$ on the boxes $G_I - n \cdot \mathbf{a}$ for all $n > 0$. We have now defined y on its trail (and perhaps on some extra lattice sites which y does not hit). The rest of the configuration can be filled in with right mirrors. From the construction it is clear that in the metric d the configuration y is periodic with period a divisor of $n_1 + n_2$. An example is given in Fig. 4. For this example $\mathbf{a}^+ = (1, 1)$, $\mathbf{a}^- = (0, -2)$, $\mathbf{a} = (1, 3)$, $n_1 = 2$, $n_2 = 4$, and the period of the motion is 6.

To see that periodic points have zero measure, consider $P_I = \{x \in X: f^I x = x\}$. The trail of each $x \in P_I$ is a periodic lift of a configuration $\hat{x}$ defined on G_I. Modifying $\mathbf{Z}^2 \backslash \mathrm{trail}(x)$ periodically with period I leads to other periodic points. For $x \in X$ periodic let $[x] := \{y \in X: \mathrm{per}(y) = \mathrm{per}(x)$ and $\mathrm{trail}(y) = \mathrm{trail}(x)\}$. The set $\mathbf{Z}^2 \backslash \mathrm{trail}(x)$ has infinite cardinality; thus, for

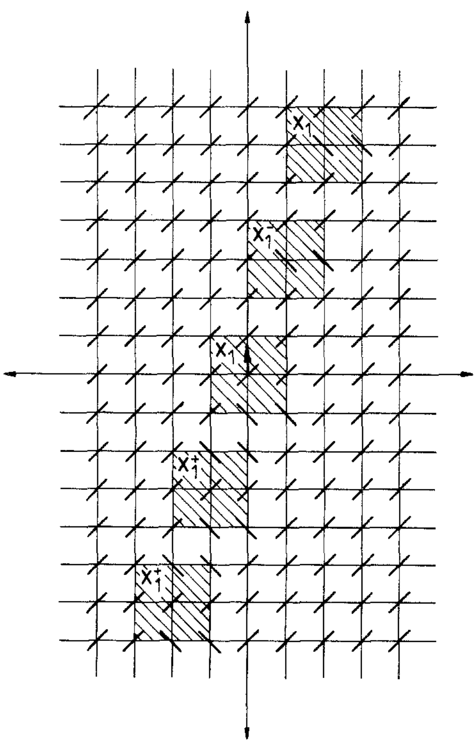

Fig. 4. A periodic point with period 6.

each $x \in P_I$ the set $[x]$ is uncountable. However, $[x] \subset \{z \in X: z_{i,j} = x_{i,j}$ if $(i, j) \in \text{trail}([x])\}$ and thus $[x]$ has measure zero.

To conclude (c), we must show that there are only a countable number of trails. To see this, notice that the set $\{[x]: \text{per}([x]) = I\}$ is a finite set, since it is determined by the motion on G_I. Thus the set $\{[x]: x \text{ periodic}\}$ is a countable set and the result follows.

(d) Consider the periodic point x of period 2 constructed in part (b) (see Fig. 3). We will show that the set $U := \{y \in X: \text{trail}(y) = \text{trail}(x)\}$ contains the full shift on M symbols for all positive M. First, clearly U is an f-invariant set. To see that U contains the full shift, we must view the set U diagonally, namely for $k \in \mathbb{Z}$ consider the lines L_k defined by the equation $i = -j + k$. The trail of x only hits the line L_{2k} at the lattice site $(2k, 2k)$ and the line L_{2k+1} at the lattice site $(2k, 2k+1)$. A point $x \in U$ defines the configuration $u^k \in \{L, R\}^{\mathbb{Z}}$ by

(i) $u^{2k} = x_{i+k,\, -i+k}$

(ii) $u^{2k+1} = x_{i+k,\, -i+k+1}$

Then $U = \{\mathbf{u} := (u_k)_{k \in \mathbb{Z}}\}$ can be though of as a set of configurations on the collection of lines L_k. Now taking into consideration the action of f on the trial of x, it is clear that the action of $f|_U$ is equivalent to the shift σ defined by

$$\sigma(\{u_i^k\}) = \{u_i^{k-1}\}$$

If $I > 0$ and $M = 2^I$, then if we consider only the first I entries of u_k (i.e., $\{L, R\}^I \subset \{L, R\}^{\mathbb{Z}}$) then it follows that (U, f) contains the full shift on M symbols. Thus $h_{\text{top}}(f) \geqslant \log M$ and since M was arbitrarily large, part (d) follows.

II. Since the cylinder sets form a basis for the topology, it is enough to check the mixing condition for $\mathscr{U}, \mathscr{V}$ cylinder sets. Fix I so large that there are cylinder sets $\mathscr{C}, \mathscr{D}$ defined on G_I such that $\mathscr{C} \subset \mathscr{U}$ and $\mathscr{D} \subset \mathscr{V}$. We can choose $n_1 > 0$ so large that if $x \in \mathscr{C}$, then $f^{n_1}x$ is outside the box G_I.[10] Likewise we choose $n_2 > 0$ so large that if $x \in \mathscr{D}$, then $\tilde{f}^{n_2}x$ is outside the box G_I. Now we construct a special $x \in \Omega$ as follows. The configuration $x \in \Omega$ will coincide with $\mathscr{C}$ on the box G_I and will coincide with $\pi_1(\tilde{f}^{n_2}\mathscr{D})$ on the box $G_I + (2I+5, 0) := \{(i, j): \max(|i - 2I - 5|, |j|) \leqslant I\}$. Now, $\mathscr{C}$ and $\mathscr{D}$ are so placed that there is room (a $5 \times I$ rectangle) to connect the lattice site where the particle left $\mathscr{C}$ to the lattice site where the particle (under $\tilde{f}$) left $\mathscr{D}$ by a simple path which does not enter the boxes G_I or $G_I + (2I+5, 0)$. If the length of the simple paths is n_3, then it is clear that $x \in \mathscr{C}$ and $f^{n_1+n_2+n_3}x \in \mathscr{D}$. The simple path can be chosen to be of length

$n_3 + i$ for any $i \geqslant 0$ simply by moving the configuration $\mathscr{D}$ to be centered at $(2I+5+i, 0)$ for all $i \geqslant 0$ and adding i empty sites (no mirrors) to the configuration to make the simple path i steps longer.

III. Define $A := \{x \in X\colon \pi_2(x) \in \{N, S\}\}$ and $B := \{x \in X\colon \pi_2(x) \in \{E, W\}\}$. Since $C = 1$, the particle turns at each vertex; thus $fA = B$ and $fB = A$. We call this preservation of parity. The proof of topological mixing is essentially the same as in II, but additionally we must preserve parity; thus, given $\mathscr{C}$, $\mathscr{D}$, the simple paths connecting them must always either be of even or odd length, depending only on $\mathscr{C}$, $\mathscr{D}$. ∎

5. PROOF OF THEOREM 2

(a) Fix a configuration x and $\delta > 0$ small. We construct a new configuration y which is periodic and within a distance δ (in the metric d_∞) of the point x. Fix I so large that (i) $18^{-I^2} < \delta$ and (ii) at least one particle of x is in the box G_I. Let x_I be the restriction of the configuration x to the box G_I. Consider the sub $\mathbf{Z}^2$ lattice $\{(i, j)\colon i = j = 0 \bmod 2I+1\}$. Tiling this sublattice with the configuration x_I gives rise to a configuration y. The motion of y is equivalent to the lift of the FM motion of x_I on G_I thought of as a torus. Since there are a finite number of particles and a finite number of states, the motion on the torus is periodic. Thus the motion of y is also periodic. The uncountability will be demonstrated in part (b).

(b) For each $I \geqslant 2$ we construct a tile x_I of size $I \times 2$ as follows:

$$\pi_1(x_I)_{0,0} = \pi_1(x_I)_{I-1,1} = L$$

$$\pi_1(x_I)_{0,1} = \pi_1(x_I)_{I-1,0} = R$$

$$\pi_1(x_I)_{i,j} = R \quad \text{for } i = 0 \bmod 2, \quad j = 0, 1$$

$$\pi_1(x_I)_{i,j} = L \quad \text{for } i = 1 \bmod 2, \quad j = 0, 1$$

$$\pi_1(x_I)_{i,j} = L \quad \text{for } i = 0 \bmod 2, \quad j = 0, 1$$

$$\pi_2(x_I)_{0,1} = S$$

$$\pi_2(x_I) = \varnothing \quad \text{otherwise}$$

Placing, for each $i \in \mathbb{Z}$, the tile x_I with left corner at lattice sites $(Ii, 0)$ and filling the rest of the lattice sites arbitrarily with mirrors gives rise to a strong periodic point with all particles having period $4I$. Figure 5 shows this example for $I = 3$. Since we can fill the rest of the lattice sites arbitrarily, the set of strong periodic points is uncountable.

Since the particles stay in an infinite horizontal strip of width 2, we can construct a configuration where all the particles have closed orbits and

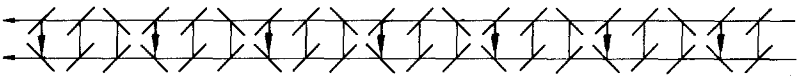

Fig. 5. A strongly periodic point.

the set of periods is $4\mathbb{Z}$. For example, this can be done by placing the tiles x_I at lattice sites $(Ii, 2I)$ for all $i \in \mathbb{Z}$. Note that these points are not strong periodic points.

(c) Given $\delta > 0$, fix I so large that $18^{-I^2} < \delta$. Now consider the following strong periodic point x: place the tiles x_2 for each $i \in \mathbb{Z}$ with left corner at the lattice site $(2i, 0)$. Fill in the rest of the lattice with left mirrors. Now construct a new point $x^* \in Z$ which is identical to x except that $\pi_1(x^*)_{I,I} = R$. The points x and x^* are distinct, yet their orbits are always closer than δ.

(d) Fix $M > 0$ large. Let $\mathscr{U} \subset Z_0$ be defined as follows: $x \in \mathscr{U}$ if and only if the following three conditions hold:

(i) $\pi_1(x_{i,j}) = \varnothing$ for all i, j

(ii) $\pi_2(x_{i,j}) = \varnothing$, $\forall j < 0$, $\forall j > M$, $\forall i$

(iii) $\pi_2(x_{i,j}) \in \{E, \varnothing\}$ for $0 \leqslant j \leqslant M$, $\forall i$

From the definition of $\mathscr{U}$ it is clear the $\mathscr{U}$ is an invariant set which is exactly the full shift on 2^M symbols. Thus the topological entropy of f on Z_0 is greater than or equal to M. Since M was arbitrary, the topological entropy is infinite. ▌

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